

REVERSIBLE AJW-ALGEBRAS

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ABSTRACT. In this article it is proved that for every special AJW-algebra A there exist central projections $e, f, g \in A$, $e + f + g = 1$ such that (1) eA is reversible and there exists a norm-closed two sided ideal I of $C^*(eA)$ such that $eA = {}^\perp({}^\perp(I_{sa})_+)_+$; (2) fA is reversible and $R^*(fA) \cap iR^*(fA) = \{0\}$; (3) gA is a totally nonreversible AJW-algebra.

1. INTRODUCTION

This article is devoted to abstract Jordan operator algebras, which are analogues of abstract W^* -algebras (AW^* -algebras) of Kaplansky. These Jordan operator algebras can be characterized as a JB-algebra satisfying the following conditions

(1) in the partially ordered set of all projections any subset of pairwise orthogonal projections has the least upper bound in this JB-algebra;

(2) every maximal associative subalgebra of this JB-algebra is generated by its projections (i.e. coincides with the least closed subalgebra containing all projections of the given subalgebra).

In the articles [6], [7] the second author introduced analogues of annihilators for Jordan algebras and gave algebraic conditions equivalent to (1) and (2). Currently, these JB-algebras are called AJW-algebras or Baer JB-algebras in the literature. Further, in [8] a classification of these algebras has been obtained. It should be noted that many of facts of the theory of JBW-algebras and their proofs hold for AJW-algebras. For example, similar to a JBW-algebra an AJW-algebra is the direct sum of special and purely exceptional Jordan algebras [8].

It is known from the theory of JBW-algebras that every special JBW-algebra can be decomposed into the direct sum of totally irreversible and reversible subalgebras. In turn, every reversible special JBW-algebra decomposes into a direct sum of subalgebra, which is the hermitian part of a von Neumann algebra and the subalgebra, enveloping real von Neumann algebra of which is purely real [5], [1]. In this paper we prove a similar result for AJW-algebras, the proof of which requires a different approach. Namely, we prove that for every special AJW-algebra A there exist central projections $e, f, g \in A$, $e + f + g = 1$ such that (1) eA is reversible and there exists a norm-closed two sided ideal I of $C^*(eA)$ such that

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$eA = {}^\perp({}^\perp(I_{sa})_+)_+$; (2) fA is reversible and $R^*(fA) \cap iR^*(fA) = \{0\}$; (3) gA is a totally nonreversible AJW-algebra.

2. PRELIMINARY NOTES

We fix the following terminology and notations.

Let \mathcal{A} be a real Banach $*$ -algebra. \mathcal{A} is called a real C^* -algebra, if $\mathcal{A}_c = \mathcal{A} + i\mathcal{A} = \{a + ib : a, b \in \mathcal{A}\}$, can be normed to become a (complex) C^* -algebra, and keeps the original norm on \mathcal{A} [9].

Let A be a JB-algebra, $P(A)$ be a set of all projections of A . Further we will use the following standard notations: $\{aba\} = U_a b := 2a(ab) - a^2 b$, $\{abc\} = a(bc) + (ac)b - (ab)c$ and $\{aAb\} = \{\{acb\} : c \in A\}$, where $a, b, c \in A$. A JB-algebra A is called an AJW-algebra, if the following conditions hold:

(1) in the partially ordered set $P(A)$ of projections any subset of pairwise orthogonal projections has the least upper bound in A ;

(2) every maximal associative subalgebra A_o of the algebra A is generated by its projections (i.e. coincides with the least closed subalgebra containing $A_o \cap P(A)$).

Let $(S)^\perp = \{a \in A : (\forall x \in S) U_a x = 0\}$, ${}^\perp(S) = \{x \in A : (\forall a \in S) U_a x = 0\}$, ${}^\perp(S)_+ = {}^\perp(S) \cap A_+$.

Then for a JB-algebra A the following conditions are equivalent:

(1) A is an AJW-algebra;

(2) for every subset $S \subset A_+$ there exists a projection $e \in A$ such that $(S)^\perp = U_e(A)$;

(3) for every subset $S \subset A$ there exists a projection $e \in A$ such that ${}^\perp(S)_+ = U_e(A_+)$ [6].

Let A be a real or complex $*$ -algebra, and let S be a nonempty subset of A . Then the set $R(S) = \{x \in A : sx = 0 \text{ for all } s \in S\}$ is called the right annihilator of S and the set $L(S) = \{x \in A : xs = 0 \text{ for all } s \in S\}$ is called the left annihilator of S . A $*$ -algebra A is called a Baer $*$ -algebra, if the right annihilator of any nonempty set $S \subseteq A$ is generated by a projection, i.e. $R(S) = gA$ for some projection $g \in A$ ($g^2 = g = g^*$). If $S = \{a\}$ then the projection $1 - g$ such that $R(S) = gA$ is called the right projection and denoted by $r(a)$. Similarly one can define the left projection $l(a)$. A (real) C^* -algebra A , which is a Baer (real) $*$ -algebra, is called an (real) AW $*$ -algebra [10], [11]. Real AW $*$ -algebras were introduced and investigated in [11], [12]. In these papers it was shown that for a real AW $*$ -algebra \mathcal{A} the C^* -algebra $M = \mathcal{A} + i\mathcal{A}$ is not necessarily a complex AW $*$ -algebra.

Let A be an AJW-algebra. By [8, Theorem 2.3] we have the equality $A = A_I \oplus A_{II} \oplus A_{III}$, where A_I is an AJW-algebra of type I, A_{II} is an AJW-algebra of type II and A_{III} is an AJW-algebra of type III [8]. By [8, Theorem 3.7] A_I , in its turn, is a direct sum of the following form

$$A_I = A_\infty \oplus A_1 \oplus A_2 \oplus \dots,$$

where A_n for every n either is $\{0\}$ or an AJW-algebra of type I_n , A_∞ is a direct sum of AJW-algebras of type I_α with α infinite. If $A = A_1 \oplus A_2 \oplus \dots$ then A is called an AJW-algebra of type I_{fin} and denoted by $A_{I_{fin}}$ and if $A = A_\infty$ then A is

called an AJW-algebra of type I_∞ and denoted by A_{I_∞} . We say that A is properly infinite if A has no nonzero central modular projection. The fact that an AJW-algebra A_{II} of type II is a JC-algebra can be proved similar to JBW-algebras [4]. Therefore, it is isomorphic to some AJW-algebra defined in [13] (i.e. to some AJW-algebra of self-adjoint operators), and by virtue of [13] $A_{II} = A_{II_1} \oplus A_{II_\infty}$, where A_{II_1} is a modular AJW-algebra of type II and A_{II_∞} is an AJW-algebra of type II, which is properly infinite. So, we have the decomposition

$$A = A_{I_{fin}} \oplus A_{I_\infty} \oplus A_{II_1} \oplus A_{II_\infty} \oplus A_{III}.$$

It is easy to verify that the part $A_{I_{fin}} \oplus A_{II_1}$ is modular, and $A_{I_\infty} \oplus A_{II_\infty} \oplus A_{III}$ is properly infinite (i.e. properly nonmodular).

3. REVERSIBILITY OF AJW-ALGEBRAS

Let A be a special AJW-algebra on a complex Hilbert space H . By $R^*(A)$ we denote the uniformly closed real $*$ -algebra in $B(H)$, generated by A , and by $C^*(A)$ the C^* -algebra, generated by A . Thus the set of elements of kind

$$\sum_{i=1}^n \prod_{j=1}^{m_i} a_{ij} (a_{ij} \in A)$$

is uniformly dense in $R^*(A)$. Let $iR^*(A)$ be the set of elements of kind ia , $a \in R^*(A)$. Then $C^*(A) = R^*(A) + iR^*(A)$. [5], [3]

Lemma 3.1. *The set $R^*(A) \cap iR^*(A)$ is a uniformly closed two sided ideal in $C^*(A)$.*

Proof. If $a, b \in R^*(A)$ and $c = id \in R^*(A) \cap iR^*(A)$, then $(a+ib)c = ac+ibid = ac - bd \in R^*(A)$. Similarly $(a+ib)c \in iR^*(A)$, i.e. $(a+ib)c \in R^*(A) \cap iR^*(A)$. Since $R^*(A) \cap iR^*(A)$ is uniformly closed and the set of elements of kind $a+ib$, $a, b \in A$ is uniformly dense in $C^*(A)$, we have $R^*(A) \cap iR^*(A)$ is a left ideal in $C^*(A)$. By the symmetry $R^*(A) \cap iR^*(A)$ is a right ideal. \triangleright

Let R be a $*$ -algebra, R_{sa} be the set of all self-adjoint elements of R , i.e. $R_{sa} = \{a \in R : a^* = a\}$.

Definition 3.2. A JC-algebra A is said to be reversible if $a_1 a_2 \dots a_n + a_n a_{n-1} \dots a_1 \in A$ for all $a_1, a_2, \dots, a_n \in A$.

Similar to JW-algebras we have the following criterion.

Lemma 3.3. *An AJW-algebra A is reversible if and only if $A = R^*(A)_{sa}$.*

Proof. It is clear that, if $A = R^*(A)_{sa}$, then A is reversible since

$$\left(\prod_{i=1}^n a_i + \prod_{i=n}^1 a_i \right)^* = \prod_{i=n}^1 a_i + \prod_{i=1}^n a_i \in R^*(A)_{sa} = A,$$

for all $a_1, a_2, \dots, a_n \in A$. Conversely, let A be a reversible AJW-algebra. The inclusion $A \subset R^*(A)_{sa}$ is evident. If $a = \sum_{i=1}^n \prod_{j=1}^{m_i} a_{ij} \in R^*(A)_{sa}$, then

$$a = \frac{1}{2}(a + a^*) = \frac{1}{2} \sum_{i=1}^n \left(\prod_{j=1}^{m_i} a_{ij} + \prod_{j=m_i}^1 a_{ij} \right) \in A.$$

Hence the converse inclusion holds, i.e. $R^*(A)_{sa} = A$. \triangleright

Lemma 3.4. *Let A be an AJW-algebra and let I be a norm-closed ideal of A . Then there exists a central projection g such that ${}^\perp({}^\perp(I_{sa})_+)_+ = gA_+$.*

Proof. Since A is an AJW-algebra there exists a projection g in A such that

$${}^\perp(I_{sa})_+ = U_{(1-g)}(A_+), {}^\perp({}^\perp(I_{sa})_+)_+ = U_g(A_+),$$

where ${}^\perp(S)_+ = \{x \in A_+ : (\forall a \in S) U_a x = 0\}$ for $S \subseteq A$.

Let (u_λ) be an approximate identity of the JB-subalgebra I and a be an arbitrary positive element in I . Then there exists a maximal associative subalgebra A_o of A containing a . Let v_μ be an approximate identity of A_o . Then $(v_\mu) \subseteq (u_\lambda)$ and $\|av_\mu - a\| \rightarrow 0$. Let $b \in A_+$ and

$$U_{v_\mu} b = 0$$

for every μ . Then $U_a U_{v_\mu} b = U_{av_\mu} b = 0$ and $U_c U_{av_\mu} b = 0$, where c is an element in A such that $b = c^2$. Hence $U_c U_{av_\mu} c^2 = 0$, $(U_c(av_\mu))^2 = 0$, $U_c(av_\mu) = 0$ and $U_c U_c(av_\mu) = U_b(av_\mu) = 0$ for every μ . We have

$$\|U_b(av_\mu) - U_b a\| = \|U_b(av_\mu) - a\| \rightarrow 0$$

because $\|av_\mu - a\| \rightarrow 0$ and the operator U_b is norm-continuous. Hence $U_b a = 0$. We may assume that $a = d^2$ for some element $d \in A$. Then

$$U_d U_b a = U_d U_b d^2 = (U_d b)^2 = 0, U_d b = 0.$$

Thus $U_d U_d b = U_{d^2} b = U_a b = 0$. Therefore, if $b \in {}^\perp((u_\lambda))_+$ then $b \in {}^\perp(I_{sa})_+$. Hence ${}^\perp((u_\lambda))_+ \subseteq {}^\perp(I_{sa})_+$. It is clear that ${}^\perp(I_{sa})_+ \subseteq {}^\perp((u_\lambda))_+$ and

$${}^\perp(I_{sa})_+ = {}^\perp((u_\lambda))_+.$$

This implies that ${}^\perp((u_\lambda))_+ = U_{(1-g)}(A_+)$ and

$$\sup_\lambda u_\lambda = g.$$

Let us prove that $U_g(A)$ is an ideal of A . Indeed, let x be an arbitrary element in A . Then $U_x u_\lambda \in I_{sa}$, i.e. $U_x u_\lambda \in U_g(A)$. By [4, Proposition 3.3.6] and the proof of [4, Lemma 4.1.5] we have U_x is a normal operator in A . Hence

$$\sup_\lambda U_x u_\lambda = U_x(\sup_\lambda u_\lambda) = U_x g.$$

At the same time

$$\sup_\lambda U_x u_\lambda \in U_g(A).$$

Hence $U_x g \in U_g(A)$. By [4, 2.8.10] we have

$$4(xg)^2 = 2gU_x g + U_x g^2 + U_g x^2 = 2gU_x g + U_x g + U_g x^2.$$

Therefore $(xg)^2 \in U_g(A)$ and $xg \in U_g(A)$.

Now, let y be an arbitrary element in $U_g A$. Then $y = U_g y$ and

$$xy = (U_g x + \{gx(1-g)\} + U_{1-g}x)U_g y = U_g x U_g y + \{gx(1-g)\}U_g y \in U_g A$$

since $\{gx(1-g)\} \in U_g A$. Hence $U_g A$ is a norm-closed ideal of A . Therefore $\{gA(1-g)\} = \{0\}$ and

$$A = U_g A \oplus U_{1-g} A.$$

This implies that g is a central projection in A and ${}^\perp({}^\perp(I_{sa})_+)_+ = gA_+$. \triangleright

Lemma 3.5. *Let A be a reversible AJW-algebra on a Hilbert space H . Then there exist two central projections e, f in A and a norm-closed two sided ideal I of $C^*(A)$ such that $e + f = 1$, $eA = {}^\perp({}^\perp(I_{sa})_+)_+$ and $R^*(fA) \cap iR^*(fA) = \{0\}$.*

Proof. Let $I = R^*(A) \cap iR^*(A)$. Since A is reversible by proposition 3.3 we have $I_{sa} \subseteq A$. By [5, 3.1] I is a two sided ideal of $C^*(A)$. Hence I_{sa} is an ideal of the AJW-algebra A . By proposition 3.4 we have there exists a central projection g such that ${}^\perp({}^\perp(I_{sa})_+)_+ = gA_+$. It is clear that g is a central projection also in $C^*(A)$.

By the definitions of I and g we have

$$R^*((1 - g)A) \cap iR^*((1 - g)A) = \{0\}.$$

\triangleright

Lemma 3.6. *Let A be an AJW-algebra and let J be the set of elements $a \in A$ such that $bac + c^*ab^* \in A$ for all $b, c \in C^*(A)$. Then J is a norm-closed ideal in A . Moreover J is a reversible AJW-algebra.*

Proof. Let $a, b \in J$, $s, t \in C^*(A)$. Then

$$s(a + b)t + t^*(a + b)s^* = (sat + t^*as^*) + (sbt + t^*bs^*) \in A,$$

i.e. J is a linear subspace of A . Now, if $a \in J$, $b \in A$, $s, t \in C^*(A)$, then

$$\begin{aligned} s(ab + ba)t + t^*(ab + ba)s^* &= (sa(bt) + (bt)^*as^*) + \\ &+ ((sb)at + t^*a(sb)^*) \in A, \end{aligned}$$

i.e. J is a norm-closed ideal of A .

Let $a_1 \in J$, $a_2, \dots, a_n \in A$ and $a = \prod_{i=2}^n a_i$. Then $a_1a + a^*a_1 \in A$ by the definition of J . Let us show that $a_1a + a^*a_1 \in J$; then, in particular, in the case of $a_2, \dots, a_n \in J$ this will imply that J is reversible. For all $b, c \in C^*(A)$ we have

$$\begin{aligned} b(a_1a + a^*a_1)c + c^*(a_1a + a^*a_1)b^* &= \\ &= (ba_1(ac) + (ac)^*a_1b^*) + ((ba^*)a_1c + c^*a_1(ba^*)^*) \in A, \end{aligned}$$

i.e. $a_1a + a^*a_1 \in J$. \triangleright

Definition 3.7. An AJW-algebra A is said to be *totally nonreversible*, if the ideal J in lemma 3.6 is equal to $\{0\}$, i.e. $J = \{0\}$.

Theorem 3.8. *Let A be a special AJW-algebra. Then there exist central projections $e, f, g \in A$, $e + f + g = 1$ such that*

- (1) $J = (e + f)A$, J is the ideal from lemma 3.6;
- (2) eA is reversible and there exists a norm-closed two sided ideal I of $C^*(eA)$ such that $eA = {}^\perp({}^\perp(I_{sa})_+)_+$;
- (3) fA is reversible and $R^*(fA) \cap iR^*(fA) = \{0\}$;
- (4) gA is a totally nonreversible AJW-algebra and

$$gA = \sum_{\omega \in \Omega} C(Q_\omega, \mathbf{R} \oplus H_\omega),$$

where Ω is a set of indices, $\{Q_\omega\}_{\omega \in \Omega}$ is an appropriate family of extremal compacts and $\{H_\omega\}_{\omega \in \Omega}$ is a family of Hilbert spaces.

Proof. We have

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_{I_\infty} \oplus A_{II_1} \oplus A_{II_\infty} \oplus A_{III}$$

and the subalgebra (without the part A_2)

$$A_1 \oplus A_3 \oplus A_4 \oplus \cdots \oplus A_{I_\infty} \oplus A_{II_1} \oplus A_{II_\infty} \oplus A_{III}$$

is reversible. The last statement can be proven similar to [4, Theorem 5.3.10]. By [14,] the subalgebra A_2 can be represented as follows

$$A_2 = \sum_{i \in \Xi} C(X_i, \mathbf{R} \oplus H_i),$$

where Ξ is a set of indices, $\{X_i\}_{i \in \Xi}$ is a family of extremal compacts and $\{H_i\}_{i \in \Xi}$ is a family of Hilbert spaces. Hence by [4, Theorem 6.2.5] there exist central projections h, g such that $A = hA \oplus gA$, hA is reversible and gA is totally nonreversible. For all $a, b_1, \dots, b_n, c_1, \dots, c_m$ in hA we have

$$b_1 \dots b_n a c_1 \dots c_m + c_m c_{m-1} \dots c_1 a b_n b_{n-1} \dots b_1 \in hA$$

since hA is reversible. Similarly for all b, c in $R^*(hA)$, $a \in hA$ we have

$$bac + c^* a b^* \in hA.$$

Hence $hA = J$

By proposition 3.5 there exist two central projections e, f in hA and a norm-closed two sided ideal I of $C^*(hA)$ such that $e + f = h$, $eA = {}^\perp({}^\perp(I_{sa})_+)_+$, fA is a reversible AJW-algebra and $R^*(fA) \cap iR^*(fA) = \{0\}$. This completes the proof. \triangleright

Let A be a special AJW-algebra. Despite the fact that for the real AW*-algebra $R^*(A)$ the C*-algebra $\mathcal{M} = R^*(A) + iR^*(A)$ is not necessarily a complex AW*-algebra we consider, that

Conjecture. Under the conditions of theorem 3.8 the following equality is valid

$$eA = I_{sa}.$$

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